Transformation of the sum and differences in product and vice versa

Formulas are:

1. $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
2. $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$
3. $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
4. $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

5. $tg\alpha \pm tg\beta = \frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}$
6. $ctg\alpha \pm ctg\beta = \frac{\sin(\alpha\pm\beta)}{\sin\alpha\sin\beta}$

7. $\sin x\cdot\cos y = \frac{1}{2}[\sin(x+y)+\sin(x-y)]$
8. $\cos x\cdot\sin y = \frac{1}{2}[\sin(x+y)-\sin(x-y)]$
9. $\cos x\cdot\cos y = \frac{1}{2}[\cos(x+y)+\cos(x-y)]$
10. $\sin x\cdot\sin y = -\frac{1}{2}[\cos(x+y)-\cos(x-y)]$

1) Transformed into a product:

a) $\sin 20^o + \cos 50^o$
b) $\sin 56^o + \cos 56^o$
c) $\sin\alpha - \sin\beta$

Solution:

a)

$\sin 20^o + \cos 50^o =$ (First we must do: $\cos 50^o = \sin 40^o$, and then we use formula…)

$$= \sin 20^o + \sin 40^o = 2\sin\frac{20^o+40^o}{2}\cos\frac{20^o-40^o}{2}$$

$$= 2\sin 30^o\cos(-10^o)$$

cos (- A) = cos A

$$= 2\sin 30^o\cos 10^o$$

$$= 2\cdot\frac{1}{2}\cos 10^o = \cos 10^o$$

b)

$$\sin 56^o + \cos 56^o =$$
$$= \sin 56^o - \sin 34^o$$
$$= 2\cos\frac{56^o + 34^o}{2}\sin\frac{56^o - 34^o}{2}$$
$$= 2\sin 45^o \sin 11^o$$
$$= 2\cdot\frac{\sqrt{2}}{2}\sin 11^o = \sqrt{2}\sin 11^o$$

c)

$$\sin\alpha - \sin\beta =$$
$$= \sin\alpha - \sin\left(\frac{\pi}{2} - \alpha\right)$$
$$= 2\cos\frac{\alpha + \frac{\pi}{2} - \alpha}{2}\sin\frac{\alpha - \left(\frac{\pi}{2} - \alpha\right)}{2}$$
$$= 2\cos\frac{\pi}{4}\sin\frac{\alpha - \frac{\pi}{2} + \alpha}{2}$$
$$= 2\cdot\frac{\sqrt{2}}{2}\sin\left(\alpha - \frac{\pi}{4}\right)$$
$$= \sqrt{2}\sin\left(\alpha - \frac{\pi}{4}\right)$$

2) Prove that

a) $\sin 15^o \sin 75^o = 0{,}25$

b) $\cos 135^o \cos 45^o = -0{,}5$

Proof:

a)

$$\sin 15^o \sin 75^o = \frac{1}{2}\left[\sin(15^o + 75^o) + \sin(15^o - 75^o)\right]$$
$$= \frac{1}{2}\left[\sin 90^o + \sin(-60^o)\right]$$
$$= \frac{1}{2}\left[\sin 90^o - \sin 60^o\right]$$
$$= \frac{1}{2}\left[1 - \frac{1}{2}\right] = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} = 0{,}25$$

b)

$$\cos 135^o \cos 45^o = \frac{1}{2}\left[\cos(135^o - 45^o) + \cos(135^o + 45^o)\right]$$

$$= \frac{1}{2}\left[\cos 90^o + \cos 180^o\right]$$

$$= \frac{1}{2}\left[0 - 1\right] = -\frac{1}{2} = -0{,}5$$

3) Solve: a)sin5xsin3x=? b) $\cos\frac{x}{2}\cos\frac{x}{3}\cos\frac{x}{4} = ?$

Solution:

a)

$$\sin 5x \sin 3x = \frac{1}{2}\left[\cos(5x - 3x) - \cos(5x + 3x)\right]$$

$$= \frac{1}{2}\left[\cos 2x - \cos 8x\right]$$

b)

$$\cos\frac{x}{2}\cos\frac{x}{3}\cos\frac{x}{4} =$$

$$= \left(\cos\frac{x}{2}\cos\frac{x}{3}\right)\cdot\cos\frac{x}{4}$$

$$= \frac{1}{2}\left[\cos\left(\frac{x}{2} - \frac{x}{3}\right) + \cos\left(\frac{x}{2} + \frac{x}{3}\right)\right]\cdot\cos\frac{x}{4}$$

$$= \frac{1}{2}\underbrace{\left(\cos\frac{x}{6}\cdot\cos\frac{x}{4}\right)}_{formula} + \frac{1}{2}\underbrace{\left(\cos\frac{5x}{6}\cdot\cos\frac{x}{4}\right)}_{formula}$$

$$= \frac{1}{2}\cdot\frac{1}{2}\left[\cos\left(\frac{x}{6} - \frac{x}{4}\right) + \cos\left(\frac{x}{6} + \frac{x}{4}\right)\right] + \frac{1}{2}\cdot\frac{1}{2}\left[\cos\left(\frac{5x}{6} - \frac{x}{4}\right) + \cos\left(\frac{5x}{6} + \frac{x}{4}\right)\right]$$

$$= \frac{1}{4}\left[\cos\frac{-x}{12} + \cos\frac{5x}{12}\right] + \frac{1}{4}\left[\cos\frac{7x}{12} + \cos\frac{13x}{12}\right]$$

$$= \frac{1}{4}\left[\cos\frac{x}{12} + \cos\frac{5x}{12} + \cos\frac{7x}{12} + \cos\frac{13x}{12}\right]$$

4) Prove that:

$$\text{a) } \sin 20^o \cdot \sin 40^o \cdot \sin 80^o = \frac{\sqrt{3}}{8}$$

$$\text{b) } \cos 10^o \cos 50^o \cdot \cos 70^o = \frac{\sqrt{3}}{8}$$

Proof: a)

$$\sin 20^o \cdot \sin 40^o \cdot \sin 80^o = \text{On first and third use formula}$$

$$= \frac{1}{2}\left[\cos 60^o - \cos 100^o\right]\sin 40^o$$

$$= \frac{1}{2}\sin 40^o\left[-\frac{1}{2} - \cos 100^o\right] \to -\cos 100^o = \cos 80^o \textit{ replace}$$

$$= \frac{1}{4}\sin 40^o + \frac{1}{2}\underbrace{\sin 40^o \cos 80^o}_{Formula}$$

$$= \frac{1}{4}\sin 40^o + \frac{1}{2}\left[\frac{1}{2}(\sin 120^o + \sin(-40^o))\right]$$

$$= \frac{1}{4}\sin 40^o + \frac{1}{4}(\sin 120^o - \sin 40^o)$$

$$= \frac{1}{4}\sin 40^o + \frac{1}{4}\sin 120^o - \frac{1}{4}\sin 40^o$$

$$= \frac{1}{4}\sin 120^o = \frac{1}{4}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8}$$

b)

This is actually the same task!

Why?

$$\cos 10^o = \sin 80^o$$

$$\cos 50^o = \sin 40^o$$

$$\cos 70^o = \sin 20^o$$

So: $\cos 10^o \cdot \cos 50^o \cdot \cos 70^o = \frac{\sqrt{3}}{8}$ INTERESTING! Is not it?

5) Transform in product: $\sin x + \sin y + \sin z,$ **if** $x + y + z = \pi$

Solution:

$$\sin x + \sin y + \sin z =$$
$$\sin x + siy + \sin[\pi - (x + y)] =$$
$$\underbrace{\sin x + \sin y}_{Formula} + \sin(x + y) =$$
$$2\sin\frac{x+y}{2}\cos\frac{x-y}{2} + 2\sin\frac{x+y}{2}\cos\frac{x+y}{2} =$$
$$2\sin\frac{x+y}{2}\underbrace{\left(\cos\frac{x-y}{2} + \cos\frac{x+y}{2}\right)}_{Formula} =$$
$$2\sin\frac{x+y}{2}\cdot 2\cdot\cos\frac{x}{2}\cdot\cos\frac{y}{2} =$$
$$4\sin\frac{x+y}{2}\cos\frac{x}{2}\cos\frac{y}{2} =$$

Transform: $\sin\frac{x+y}{2}$

$$\sin\frac{x+y}{2} = \sin\frac{\pi - z}{2} = \sin\left(\frac{\pi}{2} - \frac{z}{2}\right) = \sin\left(90^o - \frac{z}{2}\right) = \cos\frac{z}{2}$$

So: $\sin x + \sin y + \sin z = 4\cos\frac{x}{2}\cos\frac{y}{2}\cos\frac{z}{2}$

6) **Prove that:** $\sin 495^o - \sin 795^o + \sin 1095^o = 0$

Proof:

$$\sin 495^o = \sin(495^o - 360^o) = \sin 135^o = \cos 45^o$$
$$\sin 795^o = \sin(795 - 2\cdot 360^o) = \sin 75^o = +\cos 15^o$$
$$\sin 1095^o = \sin(1095^o - 3\cdot 360^o) = \sin 15^o$$

Now we have:

$$\cos 45^o - \cos 15^o + \sin 15^o =$$
$$-2\sin\frac{45^o + 15^o}{2}\sin\frac{45^o - 15^o}{2} + \sin 15^o =$$
$$-2\sin 30^o \sin 15^o + \sin 15^o =$$
$$-2\frac{1}{2}\sin 15^o + \sin 15^o = -\sin 15^o + \sin 15^o = 0$$

7) Prove that: $tg9^o - tg27^o - tg63^o + tg81^o = 4$

Proof:

$$\left(tg81^o + tg9^o\right) - \left(tg63^o + tg27^o\right) = formulas$$

$$\frac{\sin(81^o+9^o)}{\cos 81^o \cos 9^o} - \frac{\sin(63^o+27^o)}{\cos 63^o \cos 27^o} =$$

$$\frac{\sin 90^o}{\cos 81^o \cos 9^o} - \frac{\sin 90^o}{\cos 63^o \cos 27^o} = (\sin 90^o = 1)$$

$$\frac{1}{\cos 81^o \cos 9^o} - \frac{1}{\cos 63^o \cos 27^o} = \begin{pmatrix} \cos 81^o = \sin 9^o \\ \cos 63^o = \sin 27^o \end{pmatrix}$$

$$\frac{1}{\sin 9^o \cos 9^o} - \frac{1}{\sin 27^o \cos 27^o} = \left(ad\ \frac{2}{2}\right)$$

$$\frac{2}{2\sin 9^o \cos 9^o} - \frac{2}{2\sin 27^o \cos 27^o} = (\sin 2\alpha = 2\sin\alpha\cos\alpha)$$

$$\frac{2}{\sin 18^o} - \frac{2}{\sin 54^o} =$$

$$\frac{2(\sin 54^o - \sin 18^o)}{\sin 18^o \sin 54^o} =$$

$$\frac{2\cdot 2\cos\frac{54^o+18^o}{2}\sin\frac{54^o-18^o}{2}}{\sin 18^o \sin 54^o} = \frac{4\cos 36^o \cancel{\sin 18^o}}{\cancel{\sin 18^o} \sin 54^o} = \frac{4\cos 36^o}{\sin 54^o} = \frac{4\cancel{\cos 36^o}}{\cancel{\cos 36^o}} = \boxed{4}$$

8) Find $\sin 36^o$ without use of table!

Solution:

We know that:

$$\sin 36^o = \cos 54^o$$

or $$\sin 2\cdot 18^o = \cos 3\cdot 18^o$$

Next , we will use formulas: $\sin 2\alpha = 2\sin\alpha\cos\alpha$ and $\cos 3\alpha = 4\cos^4\alpha - 3\cos\alpha$

So:

$$\sin 2\cdot 18^o = 2\sin 18^o \cos 18^o$$

$$\cos 3\cdot 18^o = 4\cos^3 18^o - 3\cos 18^o$$

So:

$$4\cos^3 18^o - 3\cos 18^o = 2\sin 18^o \cos 18^o$$

(all devide with $\cos 18^o$)

$$4\cos^2 18^o - 3 = 2\sin 18^o$$

(then is: $\sin^2 18 + \cos^2 18 = 1 \Rightarrow \cos^2 18^o = 1 - \sin^2 18^o$)

$$4(1 - \sin^2 18) - 3 - 2\sin 18^o = 0$$
$$4\sin^2 18 + 2\sin 18^o - 1 = 0$$

(replacement $\sin 18^o = t$)

$$4t^2 + 2t - 1 = 0$$
$$t_{1,2} = \frac{-2 \pm \sqrt{20}}{8} = \frac{-2 \pm \sqrt{5}}{8}$$
$$t_{1,2} = \frac{-1 \pm \sqrt{5}}{4}$$
$$t_1 = \frac{-1 + \sqrt{5}}{4}$$
$$t_2 = \frac{-1 - \sqrt{5}}{4}$$
$$\sin 18^o = \frac{\sqrt{5} - 1}{4}$$

$$\cos 18^o = ?$$

$$\sin^2 18^o + \cos^2 18^o = 1$$
$$\cos^2 18^o = 1 - \left(\frac{\sqrt{5} - 1}{4}\right)^2$$
$$\cos^2 18^o = 1 - \frac{5 - 2\sqrt{5} + 1}{16}$$
$$\cos^2 18^o = \frac{16 - 6 + 2\sqrt{5}}{16}$$
$$\cos^2 18^o = \frac{10 + 2\sqrt{5}}{16}$$

$$\cos 18^o = \sqrt{\frac{10+2\sqrt{5}}{16}}$$

$$\cos 18^o = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

$$\sin 36^o = 2\sin 18^o \cos 18^o$$

$$\sin 36^o = 2 \cdot \frac{\sqrt{5}-1}{4} \cdot \frac{\sqrt{10+2\sqrt{5}}}{4}$$

$$\sin 36^o = \frac{\sqrt{\left(\sqrt{5}-1\right)^2\left(10+2\sqrt{5}\right)}}{8}$$

$$\sin 36^o = \frac{\sqrt{\left(5-2\sqrt{5}+1\right)\left(10+2\sqrt{5}\right)}}{8}$$

$$\sin 36^o = \frac{\sqrt{\left(6-2\sqrt{5}\right)\left(10+2\sqrt{5}\right)}}{8}$$

$$\sin 36^o = \frac{\sqrt{60+12\sqrt{5}-20\sqrt{5}-20}}{8}$$

$$\sin 36^o = \frac{\sqrt{40-8\sqrt{5}}}{8}$$

$$\sin 36^o = \frac{\sqrt{8\left(5-\sqrt{5}\right)}}{8}$$

$$\sin 36^o = \frac{2\sqrt{2}\sqrt{5-\sqrt{5}}}{8}$$

$$\boxed{\sin 36^o = \frac{\sqrt{2}\cdot\sqrt{5-\sqrt{5}}}{4}}$$

The end